An Analysis of the Restricted Euler Problem Using Symplectic Integrators

Henry Yip*‡[1], Jennifer M. Smillie†[1]

[1] School of Physics and Astronomy, University of Edinburgh

Open Access

Received
29 Oct 2024

Revised
17 Nov 2024

Accepted
05 Jan 2025

Published
18 Feb 2025

Abstract

The Three-Body Problem is far from fully solved despite centuries of effort. The restricted Euler Problem is a special case in which two bodies are fixed in place, resulting in two Poisson-commuting conserved quantities, allowing the system to be fully integrable by the Liouville-Arnold theorem. We analysed the restricted Euler problem using an order-4 symplectic integrator, which conserves the Hamiltonian. We used this integrator to simulate the restricted Euler problem and recovered known orbits from the literature.

DOI: 10.2218/esjs.10064 **ISSN** 3049-7930

Introduction to the Restricted Euler Problem

Consider a two-dimensional system and consider the z-axis plotted against the x-axis. The two masses m_+ and m_- are located at $(0, b)$ and $(0, -b)$ respectively whereas the third mass can move around freely (Ó'Mathúna 2008). We then try to find the trajectory of the third particle based on Newton's gravitational law. As two of the masses are stationary, this problem is also called the problem of two fixed centers.

The problem is of interest because it is one of the restricted cases of the three-body problem in which the system is fully integrable, i.e. the system is fully analytic, allowing for prediction of orbits.

We first express the system in terms of planar prolate spheroidal coordinates:

$$x = \pm\sqrt{R^2 - b^2}\sin\sigma$$
$$z = R\cos\sigma$$

where $R \geq 0$, $\sigma \in [0, \pi]$. It is similar to the polar coordinate system but with two foci. Without loss of generality, we consider $b = 1$.

Expressing Prolate Spheroidal Coordinates using Cartesian Coordinates

In this section, we will express prolate spheroidal coordinates in terms of cartesian coordinates. Knowing the relation between both systems is useful as both are commonly used. We will focus on $x > 0$. When $x < 0$, the values of R and σ correspond to those for $|x|$.

As $\cos\sigma$ is only negative in the range $\frac{\pi}{2} < \sigma < \pi$, we have $z > 0$ for $0 < \sigma < \frac{\pi}{2}$ and $z < 0$ for $\frac{\pi}{2} < \sigma < \pi$. Due to the nature of the solution, which will be apparent below, we will consider these two cases separately, and conclude by discussing $z = 0$.

*Student Author
‡Email: henry36c@gmail.com
†Corresponding academic contact: j.m.smillie@ed.ac.uk

Case for $z > 0$

Using $R = \frac{z}{\cos\sigma}$, we can substitute one of the equations into the other:

$$\begin{aligned} x &= \sin\sigma\sqrt{\left(\frac{z}{\cos\sigma}\right)^2 - 1} \\ &= \sqrt{\frac{z^2 - \cos^2\sigma}{\cos^2\sigma}}\sin\sigma \\ x^2 &= \frac{(z^2 - 1 + \sin^2\sigma)(\sin^2\sigma)}{1 - \sin^2\sigma} \\ x^2 &= \sin^2\sigma(x^2 + z^2 - 1 + \sin^2\sigma) \end{aligned}$$

where we have used $\cos\sigma = \sqrt{1 - \sin^2\sigma}$. This is only valid as $0 \leq \sigma \leq \frac{\pi}{2}$, which corresponds to $z > 0$. This is why we considered z separately.

By letting $u = \sin^2\sigma$, we have:

$$u^2 + (x^2 + z^2 - 1)u - x^2 = 0.$$

Using the quadratic formula, and noting that $\sqrt{u} = \sin\sigma$, we arrive at

$$\arcsin\sqrt{u} = \sigma_{z>0}$$

where

$$u = \frac{-(x^2 + z^2 - 1) + \sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}{2}.$$

Case for $z < 0$

We have assumed $x > 0$ and found the solution for $z > 0$. We want to find the solution for $z < 0$ while keeping $x > 0$.

Note that $\sin(\pi - \sigma) = \sin\sigma$ and $\cos(\pi - \sigma) = -\cos\sigma$. As $x \propto \sin\sigma$ and $z \propto \cos\sigma$, by letting $\sigma_{z<0} = \pi - \sigma_{z>0}$, we can find solutions for $z < 0$.

Case for $z = 0$

We have:

$$\sigma = \begin{cases} \arcsin\sqrt{u} & \text{for } z > 0 \\ \pi - \arcsin\sqrt{u} & \text{for } z < 0 \end{cases}$$

where

$$u = \frac{-(x^2 + z^2 - 1) + \sqrt{(x^2 + z^2 - 1)^2 + 4x^2}}{2}.$$

By continuity, we have $\sigma = \frac{\pi}{2}$. Another way to think of it is that this coordinate system is the polar coordinate system being stretched in the oblate direction and mirrored at the z-axis. Therefore, similar to the polar coordinate system, when $z = 0$, the angle is $\frac{\pi}{2}$.

Listing the Conserved Quantities

The conserved quantities are important to both verify theoretical models and computational results and show the system is integrable.

It is important to note that both linear and angular momentum are not conserved, as two of the bodies are fixed in place, implying the existence of external forces. This can be illustrated by releasing the third body a distance away from the two fixed bodies. The initial linear and angular momentum are zero, but the third body is attracted by the two fixed bodies and accelerates towards them.

We define two constants, g and h, below (Dullin *et al.* 2016):

$$g = \frac{1}{2}\left(zp_x - xp_z\right)^2 + \frac{1}{2}b^2p_z^2 + bz\left(\frac{m_1}{\sqrt{(z+b)^2+x^2}} - \frac{m_2}{\sqrt{(z-b)^2+x^2}}\right)$$
$$h = \frac{1}{2}\left(p_z^2+p_x^2\right) - \frac{m_1}{\sqrt{(z+b)^2+x^2}} - \frac{m_2}{\sqrt{(z-b)^2+x^2}}$$

where b, as defined above, is the distance of the two centers from the origin, and p_x and p_z are momentum in the x and z directions, respectively. We take $b = 1$, and therefore:

$$g = \frac{1}{2}\left(zp_x - xp_z\right)^2 + \frac{1}{2}p_z^2 + bz\left(\frac{m_1}{\sqrt{(z+1)^2+x^2}} - \frac{m_2}{\sqrt{(z-1)^2+x^2}}\right)$$
$$h = \frac{1}{2}\left(p_z^2+p_x^2\right) - \frac{m_1}{\sqrt{(z+1)^2+x^2}} - \frac{m_2}{\sqrt{(z-1)^2+x^2}}$$

where g is a symmetry in the phase space, and h is the energy of the third particle. It can also be shown that g is a first integral of motion and the two constants commute under the Poisson bracket. Hence, the system is integrable by the Liouville-Arnold theorem.

Choosing an Integration Method for Numerical Simulation

Below, we aim to choose an integration method that can be used for a general three-body problem, so we assume all three bodies can move freely. As we are integrating a Liouville integrable system, we choose symplectic integrators, which conserve the Hamiltonian. This is discussed extensively in Casey (2020), which we will use for the whole section.

Starting from $i = 1$, we denote each particle by m (m ranges from 1-3). Using x and v to represent positions and velocities, respectively, we first run:

$$v_m^{(i)} = v_m^{(i-1)} + c_i a(x_m^{(i-1)})dt$$

for each m, in order of $m = 1$, $m = 2$, then $m = 3$. c_i is an integer, which is further explained below. dt is the step size in the numerical simulation. After this step, we run:

$$x_m^{(i)} = x_m^{(i-1)} + d_i v_m^{(i)} dt$$

for each m, as above. We have d_i as an integer. After these two steps are run, we increase i by 1. This process repeats until i reaches n, where n is defined as the order of the symplectic integrator. The whole process is illustrated in the example below, corresponding to $n = 2$:

Particle 1, Steps 1-6

$$v_1^{(1)} = v^{(0)} + c_1 a(x^{(0)})dt$$
$$x_1^{(1)} = x^{(0)} + d_1 v^{(1)} dt$$
$$v_2^{(1)} = v^{(0)} + c_1 a(x^{(0)})dt$$
$$x_2^{(1)} = x^{(0)} + d_1 v^{(1)} dt$$
$$v_3^{(1)} = v^{(0)} + c_1 a(x^{(0)})dt$$
$$x_3^{(1)} = x^{(0)} + d_1 v^{(1)} dt$$

Particle 2, Steps 7-12

$$v_1^{(2)} = v^{(1)} + c_2 a(x^{(1)})dt$$
$$x_1^{(2)} = x^{(1)} + d_2 v^{(2)} dt$$
$$v_2^{(2)} = v^{(1)} + c_2 a(x^{(1)})dt$$
$$x_2^{(2)} = x^{(1)} + d_2 v^{(2)} dt$$
$$v_3^{(2)} = v^{(1)} + c_2 a(x^{(1)})dt$$
$$x_3^{(2)} = x^{(1)} + d_2 v^{(2)} dt$$

For compactness, we can put all the c_i and d_i into a vector:

$$\mathbf{c} = [c_1, c_2, ..., c_{n-1}, c_n]$$
$$\mathbf{d} = [d_1, d_2, ..., d_{n-1}, d_n].$$

Furthermore, we can put $\mathbf{c}$ and $\mathbf{d}$ into a matrix:

$$\mathbf{A} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}.$$

Types of Symplectic Integrators

It has been shown by Casey (2020), firstly, that

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

leads to a first-order symplectic integration method; this is the Euler method. Meanwhile,

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

leads to a second-order symplectic integration method; the Verlet method. Furthermore,

$$\mathbf{A} = \begin{bmatrix} \frac{7}{24} & \frac{3}{4} & \frac{-1}{24} \\ \frac{2}{3} & \frac{-2}{3} & 1 \end{bmatrix}$$

leads to a third-order symplectic integration method, referred to as the Ruth method. And lastly,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2\left(2-2^{1/3}\right)} & \frac{1-2^{1/3}}{2\left(2-2^{1/3}\right)} & \frac{1-2^{1/3}}{2\left(2-2^{1/3}\right)} & \frac{1}{2\left(2-2^{1/3}\right)} \\ \frac{1}{2-2^{1/3}} & \frac{-2^{1/3}}{2-2^{1/3}} & \frac{1}{2-2^{1/3}} & 0 \end{bmatrix}$$

leads to a fourth-order symplectic integration method, which will be referred to as the Neri method. In theory, a n^{th} order symplectic method is supposed to have an error of $O(h^n)$, where h is the step size of the system. Therefore, the error should drop as the order of the method increases.

Metrics for Choosing an Integration Method

We used several metrics to evaluate the precision of the numerical methods specified above. As we built this code with the purpose of simulating the general three-body problem, these metrics were compared across three different orbits, namely the Figure-8 orbit, the Bumblebee orbit, and the Moth orbit. Each orbit was simulated with a total of $100,000$ steps, using a step size of 0.0001.

The first metric compares the error of energy to the initial energy, which can be named as energy deviation:

$$\frac{\Delta E}{E_0} = \left| \frac{E_{\max} - E_{\min}}{E_0} \right|$$

where E_0 is the energy at the start of the simulation.

The second metric is named the momentum difference. For many systems, the initial momentum is zero. Therefore, a momentum deviation cannot be well defined. Instead, we will only check the momentum difference, which can be defined as:

$$\text{Max } \Delta p_x = |p_{\max,x} - p_{\min,x}|$$
$$\text{Max } \Delta p_y = |p_{\max,y} - p_{\min,y}|.$$

Method	Orbit	Energy Deviation	Max Δp_x	Max Δp_y	Run Time (s)
Neri (Order 4)	Figure-8	6.124×10^{-14}	1.040×10^{-13}	4.852×10^{-14}	29.60
Ruth (Order 3)	Figure-8	1.280×10^{-13}	5.307×10^{-14}	3.048×10^{-14}	25.91
Verlet (Order 2)	Figure-8	5.893×10^{-9}	3.841×10^{-14}	2.232×10^{-14}	12.31
Euler (Order 1)	Figure-8	3.601×10^{-5}	3.686×10^{-14}	3.009×10^{-14}	15.53
Neri (Order 4)	Bumblebee	8.058×10^{-3}	4.591×10^{-14}	9.246×10^{-14}	27.79
Ruth (Order 3)	Bumblebee	8.058×10^{-3}	7.394×10^{-14}	4.249×10^{-14}	23.84
Verlet (Order 2)	Bumblebee	9.676×10^{-2}	2.287×10^{-14}	1.882×10^{-14}	11.56
Euler (Order 1)	Bumblebee	2.4329	3.625×10^{-14}	2.312×10^{-14}	15.71
Neri (Order 4)	Moth	1.459×10^{-9}	4.874×10^{-14}	8.693×10^{-14}	28.06
Ruth (Order 3)	Moth	2.143×10^{-8}	3.486×10^{-14}	3.185×10^{-14}	22.39
Verlet (Order 2)	Moth	4.775×10^{-5}	2.387×10^{-14}	3.835×10^{-14}	11.55
Euler (Order 1)	Moth	0.02070	2.031×10^{-14}	2.183×10^{-14}	13.76

Table 1: Comparison of Ruth, Neri, Verlet, and Euler methods across different orbits.

The third metric is the run time. In our scenario, most simulations are relatively short, so it is not as important as the metrics above. Below all the metrics are compared for each orbit.

First, the Euler method has the highest energy inaccuracy by far, while having a higher running time than the Verlet method. The latter result is surprising, and is possibly due to the 0 entry in the d_2 corresponding to the Verlet method. Besides, it can be seen that the Neri method conserves energy much better than other methods, especially in the Moth orbit simulation. More interestingly, the Verlet method conserves momentum slightly better than methods with higher orders. However, the difference is too small to be significant (all of order 10^{-14}), and may not be accurate, as they are close to the machine error for double-precision floating-point numbers in Python, which is $O(10^{-16})$. Even though the Neri method causes a longer run time, the difference is negligible. Therefore, the Neri method is chosen.

Results

Types of Orbits

We have reproduced three types of orbits as described in Dullin *et al.* (2016). First, we reproduced the satellite orbit, in which the z-coordinate of the third body never changes sign. In other words, the third body never seems to pass through the mid-point between the two fixed centers. Next, we reproduced the planetary orbit, in which the third body almost forms a complete ellipse around the two centers. Finally, we reproduced the lemniscate orbit, in which the third body passes through the line connecting the two fixed centers, unlike both satellite and planetary orbits. The trajectories of the orbits are included in the GitHub repository available at the end of this paper.

Predicting Type of Orbit from Integrals of Motion

Referring to g and h, as defined above, the value of h must be negative, as we are solving a bounded system, but the values for g greatly vary. It has been shown in literature (Dullin *et al.* 2016) that g and h, the integrals of motion shown above, can predict the type of orbit. In general, negative g corresponds to a satellite orbit, a slightly positive g (say, between 0 and 1) corresponds to a lemniscate orbit, and a larger g corresponds to a planetary orbit. This is different for different combinations of m_+ and m_-, but we only considered the symmetric case. We have verified that the satellite orbit corresponds to $g < 0$ and $h < 0$, the lemniscate orbit corresponds to $0 < g < 1$ and $-1 < h < 0$, and the planetary orbit corresponds to $g > 1$ and $-0.5 < h < 0$, which agrees with literature (Dullin *et al.* 2016).

Textbook Verification by Comparing Eccentricities and Semi-major Axes

To conclude, we can compare with the constants defined by Ó'Mathúna (2008). It is defined that M and N are constants, where

$$\frac{1}{2}\frac{\left(R^2 - b^2\cos^2\sigma\right)^2}{R^2 - b^2}\dot{R}^2 = ER^2 + \mu R + M$$
$$\frac{1}{2}\left(R^2 - b^2\cos^2\sigma\right)^2\dot{\sigma}^2 = -Eb^2\cos^2\sigma + \frac{(m_+ - m_-)\mu b\cos\sigma}{m_+ + m_-} + N.$$

We have $\mu = G(m_1 + m_2)$ and G is the gravitational constant between particles, which is usually taken as 1 for simplicity.

It is stated in the textbook that when $\eta^2 < 1$, it corresponds to a closed elliptic orbit of eccentricity η and semimajor axis p. The two constants are defined as:

$$p = \frac{O^2}{2}$$
$$\eta = \frac{1}{p}$$

where $O = \sqrt{2N}$ is a constant with units of angular momentum. In our code, only the planetary orbit satisfies $\eta^2 < 1$. This agrees with theoretical results, as the planetary orbit is the only type of closed orbit.

To verify our estimate of the semi-major axis p is accurate, we have also measured the semi-major axis of a planetary orbit directly from its trajectories in the x-z plane. If the code is accurate, using $\eta = \frac{1}{p}$, we should obtain $p = 2.35$ if we use an orbit with $\eta = 0.425$. Instead, we obtained $p = 2.53$ with a standard deviation of 0.133, showing a small discrepancy. This inaccuracy should be investigated over a range of η values.

Conclusion

We selected an integration method and used it to analyze Euler's Three Body Problem. We demonstrated that the three types of orbits in the Euler Problem can be recreated. Further work could involve simulating and exploring different cases discussed by Ó'Mathúna (2008), as well as comparing results with other integrator types, such as regularized integrators and adaptive step size integrators.

Data Availability

The code used in this project, as well as some additional diagrams are available on GitHub: `https://github.com/Henry-Yip/Three_Body_Problem_Code`.

Acknowledgements

I would like to express my gratitude to Prof. Jenni Smillie for generously taking the time to supervise this project. I am also thankful to the School of Physics and Astronomy, University of Edinburgh for funding this project through the Career Development Scholarship. Additionally, I would like to thank Allison Lau (University of Toronto) for her encouraging comments.

References

Casey, R. M. 'Computer Implementation of Symplectic Integrators and Their Applications to the N-body Problem' (2020)

Dullin, H. R. and Montgomery, R. 'Syzygies in the Two Center Problem' Nonlinearity **29** 4 (2016)

Ó'Mathúna, D. 'Integrable Systems in Celestial Mechanics' (Springer Science & Business Media; 2008)