Investigating the Inharmonicity of Piano Strings

Eleanor Roy*[1]

[1] School of Physics and Astronomy, University of Edinburgh

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Abstract

In classical physics, ideal string vibrations are modelled using the harmonic series, yet real mode frequencies deviate from the integer set of harmonics due to string stiffness, known as inharmonicity. This work investigates this phenomenon in five piano strings on a Yamaha GB1 Grand Piano. Using the audio software *Audacity* for spectral analysis, the inharmonicity coefficient is determined experimentally and theoretically based on string properties. These values are comparable with previous research, offering insights into tuning techniques to minimise dissonant inharmonicity. Possible innovations from inharmonicity research are explored, with suggestions such as the temperature-dependent self-tuning systems for acoustic pianos.

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Introduction

Rationale

Piano strings are tuned to account for inharmonicity, a deviation from harmonic frequencies caused by string stiffness (Heetveld *et al.* 1984). Understanding these patterns informs tuning strategies to improve sound quality. Advancing inharmonicity research could further modernise acoustic pianos, revolutionising music and science.

Mode Frequencies & Inharmonicity

The fundamental frequency is the natural frequency at which an object vibrates when struck or plucked, producing the lowest and most dominant pitch in the sound it generates. It is the lowest frequency of a vibrating object, with harmonics being exact integer multiples of this frequency, forming the harmonic series. Piano strings, however, exhibit resonant frequencies that deviate from these harmonics, known as in-harmonic partials (Nave 2020). This deviation occurs because the theory of harmonic frequencies assumes ideal strings with no stiffness, which is never the case in the real world. The degree of this deviation is called inharmonicity (Cohen 1984).

Inharmonicity, resulting from string stiffness, is crucial in tuning. For example, the sixteenth harmonic, equivalent to four octaves above another note, is approximately a semitone higher than an ideal string's harmonic due to the progressive sharpening of mode frequencies. This sharpening means that the note has a higher frequency than the original. The effect is influenced by the string's diameter-to-length ratio (Shankland *et al.* 1939; Berg 2020). A higher ratio leads to greater sharpening because increased stiffness from a thicker string causes the vibrational frequencies to deviate further from their ideal harmonic values.

Controversially, a small amount of inharmonicity may be desirable for the piano's distinctive sound, as it adds warmth and richness to the tone, contributing to the instrument's characteristic timbre. However, excessive inharmonicity can degrade sound quality by making the pitch unclear and less harmonious (Campbell *et al.* 1994).

*Student Author

Theory

Harvey Fletcher's research on piano string inharmonicity has been fundamental in acoustics (Fletcher 1964; Pierce 1983). Fletcher's equations remain highly influential and widely applied in the field today.

Fletcher derived Equation 1 for the frequency, f_n, of the n^{th} mode of a string by considering the equation of motion of a circular string fixed at either end, accounting for the tension and elastic stiffness causing a restoring force, and the principles of energy conservation in the bending and stretching of the string:

$$f_n = nF(1 + Bn^2)^{1/2} \qquad \text{for} \quad n = 1, 2, 3, ... \tag{1}$$

where F and B are two constants that can be obtained from an accurate measurement of the frequencies of any two modes. In particular, B is the inharmonicity coefficient with units of m^{-2}s^2.

In this paper, Equation 1 will be used to calculate the experimental inharmonicity coefficient. Although specific literature values are unavailable, the theoretical and experimental data from Fletcher (1964) on an upright Hamilton piano will provide a critical reference for comparison.

To calculate the theoretical inharmonicity coefficient, B, we use the equations from Fletcher (1964) for solid-steel and copper-wound steel strings, considering relevant physical properties. The analysis includes three copper-wound steel strings (named C1–C3) and two solid-steel strings (C4 & C5) which are discussed in more detail later. For steel strings, the formula is:

$$B = 3.95 \times 10^{10} \left(\frac{d^2}{l^4 {f_0}^2} \right) \tag{2}$$

where d is the diameter, l is the length, and f_0 represents fundamental frequency for the first harmonic. For copper-wound steel strings, a similar approach applies. Assuming only the steel core contributes to the inharmonicity, the formula becomes:

$$B = 4.6 \times 10^{10} \left(\frac{d^4}{D^2 l^4 {f_0}^2} \right) \tag{3}$$

with d the inner steel diameter, and D the outer diameter including the copper winding.

In the real world, f_0 and F can be approximated as nearly equal at the fundamental mode because the effects of inharmonicity are minimal at the lowest frequency — the impact of string stiffness is relatively small, meaning the actual frequency, F, closely matches the theoretical harmonic frequency, f_0. C1–C5 have relatively low fundamental frequencies (inharmonicity is most pronounced at higher modes), meaning the difference between f_0 and F is negligible. These calculations estimate inharmonicity for both string types, enabling comparison with experimental values.

Methods

On a piano, C1 to C5 correspond to C notes across different octaves: C1 is the lowest, in the deep bass range; C2 and C3 are higher in the bass; C4 is Middle C, the central reference; and C5 is one octave above Middle C in the mid-treble range. C1–C3 are copper-wound strings with an inner diameter, d, outer diameter, D, and length, l, while C4 & C5 are steel-only strings, having only a diameter, d, and length, l.

After a preliminary analysis of strings C1–C8, the first five strings (C1–C5) were selected for their clearly identifiable modes, with frequency peaks most easily visualised using *Audacity* software for accurate frequency determination.

The inharmonicity coefficient, B, was determined using two complementary methods: experimental and theoretical. The experimental method involved measuring the frequencies, f_n, across multiple modes and calculating B via Equation 1. This was done by plucking the strings, recording frequencies with *Audacity*, and performing Fourier analysis to decompose the recorded sound signal into its constituent frequencies, allowing for the identification of fundamental and overtone frequencies. Based on these data points, the inharmonicity coefficient is calculated for both solid-steel and copper-wound strings.

In the theoretical approach, physical properties (diameter, d, length, l, and outer diameter, D, for copper-wound strings C1–C3) were measured, and Equations 2 & 3 used to calculate B. However, even in this method, the fundamental frequency, f_0, must be experimentally obtained. Thus, the theoretical approach, while relying on physical measurements, still depends on f_0 for the calculation.

Equation 1, derived by Fletcher, relates the frequencies of different vibration modes to the string's physical properties to determine B. Measurements of string length, diameter, and frequency were used, and linear regression was applied to a transformed version of this equation to calculate B.

Results & Discussion

As seen in Figure 1, the experimental values align closely with the theoretical values for most strings, exhibiting minimal deviation. For string C5, the discrepancy is slightly higher, likely due to measurement uncertainties, factors affecting higher frequencies, and increased inharmonicity from greater stiffness and tension in shorter, higher-pitched strings. Additionally, Equation 2 shows that the terms are to the power of 4, meaning any small measurement discrepancies in length or frequency can result in significant errors in the calculation.

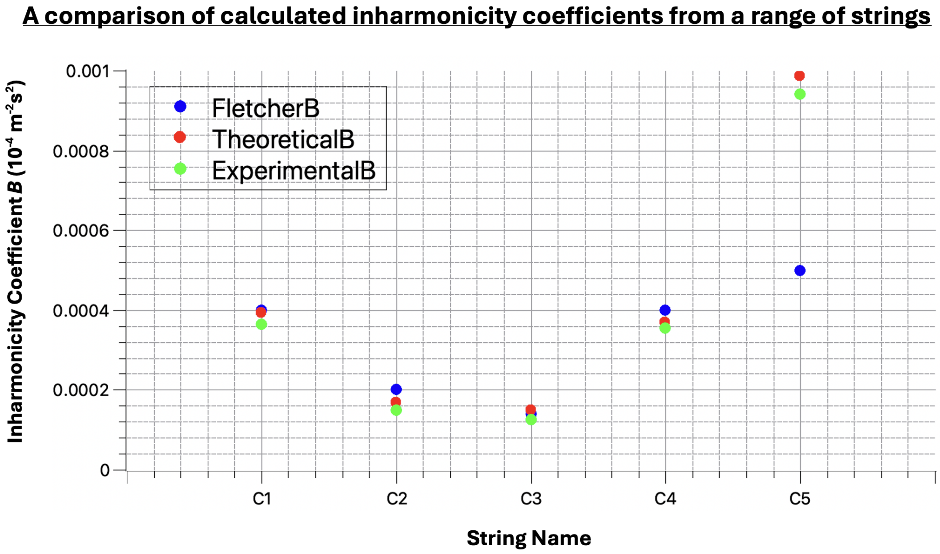

Figure 1: Graph showing experimental and theoretical inharmonicity coefficients as well as those taken from Fletcher (1964). *NB:* error bars are too small to be seen in the image.

These findings indicate a trend of increasing inharmonicity for steel-only strings (C4 & C5) as string numbers rise, attributed to progressive sharpening of partials with increasing mode numbers. Conversely, inharmonicity decreases for copper-wound strings (C1–C3), due to the decreasing thickness of the copper winding, forming the characteristic shape shown in Figure 1.

The graph provides a comparison with Fletcher's values, serving as a key benchmark. Despite deriving from a different piano over 60 years ago, the experimental data aligns well with these results. Variations are expected due to differences in piano construction, string materials, and lengths. Nonetheless, the trend of higher inharmonicity in shorter and thicker strings remains consistent. As string number increases, string length decreases, leading to greater stiffness in higher-numbered strings, while the lower, thicker strings (due to the copper-winding) exhibit increased inharmonicity due to their larger diameter.

The deviation observed in Fletcher's predicted value for C5, which is lower than this result, suggests differences in the energy distribution along the string during vibration, especially at higher frequencies. As strings vibrate, energy distributes across multiple modes, but in shorter, higher-pitched strings like C5, increased stiffness and tension confine energy to the string's boundaries. This confinement leads to more pronounced inharmonicity, as energy transfer becomes less efficient, resulting in a frequency shift. Additionally, variations in string construction, such as winding density, material properties, or tension adjustments, can alter vibrational responses, affecting inharmonicity measurements. Despite these differences, the overall consistency with Fletcher's findings reinforces the reliability of this study, even when accounting for variations in instruments and experimental conditions.

Conclusions

This work investigated the inharmonicity coefficient of five piano strings using both theoretical and experimental methods. The results showed consistency with each other and aligned with Fletcher's findings. A key observation was that higher strings exhibit increased inharmonicity due to the progressive sharpening of partials with increasing mode number, influenced by the string's diameter-to-length ratio (Shankland *et al.* 1939), which rises as string length decreases.

Historically, inharmonicity was overcome by tuning octaves by ear. Today, advancements in technology and a deeper understanding of inharmonicity have led to electronic tuning devices, paving the way for self-tuning acoustic pianos. With further development, temperature-dependent self-tuning systems could be globally commercialised, enabling modern mechanisms to correct harmonic deviations and establishing self-tuning acoustic pianos as instruments of the future.

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References

Berg, R. E. 'Sound' *Accessed 4th Feb 2021* (2020)
Campbell, M. and Greated, C. 'The Musician's Guide to Acoustics' (Oxford University Press; 1994)
Cohen, E. A. 'Some Effects of Inharmonic Partials on Interval Perception' Music Perception **1** 3 (1984)
Fletcher, H. 'Normal Vibration Frequencies of a Stiff Piano String' The Journal of the Acoustical Society of America **36** 1 (1964)
Heetveld, V. and Rasch, R. A. 'String Inharmonicity and Piano Tuning' The Journal of the Acoustical Society of America **76** S1 (1984)
Nave, R. 'Fundamental and Harmonics' *Accessed 2nd Feb 2021* (2020)
Pierce, J. R. 'The Science of Musical Sound' (Scientific American Library; 1983)
Shankland, R. S. and Coltman, J. W. 'The Departure of the Overtones of a Vibrating Wire From a True Harmonic Series' The Journal of the Acoustical Society of America **10** 3 (1939)